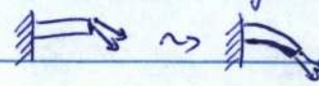
Mechanics

- notes online: http://www.cims.nyu.edu/~wirth/mechanics.html
- weekly homework online (bring to lecture)
- basic outline: – elasticity (~ 7 weeks)
  – classical mechanics (~ 4 weeks)
  – statistical mechanics (~ 3 weeks)

Elasticity

- standard references:
  - Landau, Lifschitz: Theory of Elasticity
  - Marsden, Hughes: Mathematical Foundations of Elasticity
  - Ciarlet: Mathematical Elasticity
- brief existence theory: • Pedregal: Variational Methods in Nonlinear Elasticity
- many applications/examples:
  - Howell, Kozyreff, Ockendon: Applied Solid Mechanics
  - Antman: Nonlinear Problems of Elasticity

Elasticity describes the deformation of solid materials and the associated forces.
Basic questions: How does solid deform under a load?
What are the internal material forces?



nonlinear theory (large deformations)
linearized theory (small deformations)

__Brief history__ (from my notes during a lecture by John Ball)

1678 Hooke's law
1705 Jakob Bernoulli }
1742 Daniel Bernoulli } Elastica (elastic rod)
1744 Leonard Euler }
1821 Navier, special case of linear elasticity via molecular model
1822 Cauchy, stress, nonlinear & linear elasticity
forgotten for a long time
1927 Love, Treatise on linear elasticity
1950s Ronald Rivlin, Exact solutions in incompressible nonlinear elasticity
1960–1980 nonlinear theory developed by Ericksen, Truesdell, ...
1980s Mathematical developments, applications to materials science

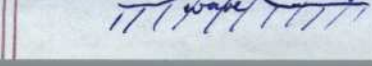
__Key elements__

kinematics: description of (the geometry of) motion; κίνημα = motion
kinetics/dynamics: description of forces and their relation to motion; δύναμις = force
statics: description of static equilibrium

Associated concepts: strain (local description of material distortion)
stress (local description of material forces)
balance laws (conservation of mass, linear & angular momentum)
constitutive laws (relation between stress & strain, material property)

We start with kinematics, including the description of strain.

__Eulerian/spatial coordinates__: Fix a point y in space and study how physical quantities at y change over time (e.g. velocity $v(y,t)$ or density $\rho(y,t)$). Over time, y is occupied by different particles. Often used in fluid dynamics, because resulting PDEs have simple form, but difficult to use with free boundaries:



$\cdot y$ (sometimes occupied by water, sometimes by air)

Lagrangian / material coordinates: Fix a particle and track it over time.
Particles are labeled by their position x in a reference configuration (often initial or unloaded configuration, but in fact need not be occupied at any time).
This is what we use in (nonlinear) elasticity, since free boundaries are handled automatically (in linearized elasticity one assumes so small deformations that a particle position stays approx. constant so that Lagrangian and Eulerian coords. are identical).

In the reference configuration we assume the solid to occupy a sufficiently smooth open connected region $\Omega \subset \mathbb{R}^3$ (a "domain").

reference configuration (Lagrangian coord.) deformed configuration (Eulerian coord.)

e.g. particle position $y : \Omega \times [0,T] \to \mathbb{R}^3$
or material density $\rho : \Omega \times [0,T] \to \mathbb{R}$

reason for choosing Ω open: contact possible 

Strain

deformation gradient: $F = Dy = \begin{pmatrix} \partial y_1/\partial x_1 & - & \partial y_1/\partial x_3 \\ \partial y_3/\partial x_1 & - & \partial y_3/\partial x_3 \end{pmatrix}$

When is a field $F : \Omega \to \mathbb{R}^{3\times 3}$ a deformation gradient?

F is a gradient ("conservative vector field")
$\Leftrightarrow$ its line integrals are path-independent
$\Leftrightarrow$ line integrals along closed curves are zero
$\underset{\text{if } \Omega \text{ is simply connected}}{\Leftrightarrow}$ $\operatorname{curl} F = \begin{pmatrix} \operatorname{curl}(Dy_1) \\ \vdots \\ \operatorname{curl}(Dy_3) \end{pmatrix} = 0$

right Cauchy-Green deformation tensor: $C = F^T F$ (left version is $B = FF^T$)

When is a field $C : \Omega \to \mathbb{R}^{3\times 3}$ a right Cauchy-Green def. tensor?

$F^T F$ is symm. pos. def. and thus a Riemannian metric on $\mathbb{R}^3$,
$|dx|^2_{F^T F} = \langle F^T F dx, dx \rangle = |F dx|^2$.
This is $|dy|^2$ iff F is a def. grad., and then $C = F^T F$ is the standard Euclidean metric in the deformed configuration, expressed in Lagrangian coordinates. Hence, C is right Cauchy-Green def. tensor
$\Leftrightarrow$ C is a flat metric
$\underset{\text{locally}}{\Leftrightarrow}$ $\underbrace{\text{Riemann curvature tensor}}_{\text{depends on } 1^{st} \,\&\, 2^{nd} \text{ derivatives of } C}$ vanishes

We assume $y(\cdot,t)$ to be orientation-preserving, i.e. $J = \det F(x,t) > 0$ for a.e. $x \in \Omega$, and invertible on Ω

Inverse function thm: $J > 0 \Rightarrow y(\cdot,t)$ is locally invertible.

Global inverse fcn thm (Ball, 1981): Let bdd domain $\Omega \subset \mathbb{R}^n$ have smooth boundary $\partial\Omega$, $y \in C^1(\bar{\Omega}; \mathbb{R}^n)$, $\det Dy > 0$ on $\bar{\Omega}$, $y|_{\partial\Omega}$ injective. Then y is invertible on $\bar{\Omega}$.
(proof via Brouwer degree; weaker version with $y \in W^{1,p}(\Omega; \mathbb{R}^n)$, $p > n$, $\partial\Omega$ Lipschitz, $y(\bar{\Omega})$ satisfying a cone condition, $\int_\Omega |Du^{-1}|^q \det Du \, dx < \infty$ for $q > n$)

Analysis of material distortion and strain

Square root thm: Let $C \in \mathbb{R}^{n\times n}$ be symm. pos. def.. There is a unique sym. pos. def. $U \in \mathbb{R}^{n\times n}$ with $C = U^2$ ("$U = \sqrt{C}$").

proof: existence: Spectral decomposition yields $C = R^T \Lambda R$ for $R \in SO(n)$ and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ with eigenvalues $\lambda_1, \dots, \lambda_n > 0$.
$U = R^T \sqrt{\Lambda} R$ satisfies $C = U^2$ with $\sqrt{\Lambda} = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$.

uniqueness: Let $C = U^2 = V^2$ with U, V sym. pos. def., and let v be an eigenvector of C with eigenvalue λ, then
$0 = (U^2 - \lambda I)v = (U + \sqrt{\lambda} I)\underbrace{(U - \sqrt{\lambda} I)v}_{=:w}$, hence $Uw = -\sqrt{\lambda}\, w$ so that $w = 0$.
Thus $Uv = \sqrt{\lambda}\, v$. Likewise $Vv = \sqrt{\lambda}\, v$. Since the eigenvectors of C span $\mathbb{R}^n$, U and V must coincide. □

Polar decomposition thm (Cauchy): If $\det F > 0$, then there exist unique matrices $R \in SO(n)$ and U, V sym. pos. def. such that $F = RU = VR$.

proof: Choose $U = \sqrt{F^T F}$, $R = FU^{-1}$, $V = RUR^T$ ($R \in SO(n)$ follows from $\det R = \det F (\det U)^{-1} > 0$ and $R^T R = U^{-1} U^2 U^{-1} = 0$).
Uniqueness follows since $F = RU = VR$ implies $U^2 = F^T F$ and $V^2 = FF^T$. □

Note:
- $y(x+z, t) = y(x,t) + F(x,t)z + o(|z|)$ so that locally the deformation can be described by the deformation gradient F (its linearization).
- A deformation $y(x) = Ux$ with U sym. pos. def. (i.e. $U = Q^T \Lambda Q$, $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, $Q = (v_1 | \dots | v_n) \in SO(n)$) is a stretching/compression along the (orthogonal eigenvectors $v_1, \dots, v_n$ by the factor $\lambda_1, \dots, \lambda_n$.
  Hence, a deformation with def. grad. F can locally be interpreted as composition of a stretching U and a rotation R or of a rotation R and a stretching V
- Using $U = Q^T \Lambda Q$ we can write $F = (RQ^T)\Lambda Q$, which is the singular value decomposition of F, where $\lambda_1, \dots, \lambda_n$ are the singular values of F.
  Hence, a deformation with def. grad. F can locally be interpreted as a composition of a rotation Q, a stretching Λ along the coordinate axes by $\lambda_1, \dots, \lambda_n$, and a rotation (RQ^T).

Since rotations do not involve material deformations (with internal forces and energy costs), the important distortion information is fully captured by C or equivalently by the Lagrangian strain $E = \frac{1}{2}(C - I)$. In fact, one can reduce the representation even further to just three numbers, which are invariant under composing F with rotations from the left or right:

Strain invariants (fully describe amount of stretching / compression)

principal stretches: singular values of F

matrix invariants I_C, II_C, III_C: $\det(C - \lambda I) =: -\lambda^3 + I_C \lambda - II_C \lambda^2 + III_C$

If $\lambda_1, \lambda_2, \lambda_3$ are principal stretches ($\lambda_1^2, \lambda_2^2, \lambda_3^2$ are eigenvals of C), then $\det(C - \lambda I) = \det\left(Q^T\left(\begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix} - \lambda I\right)Q\right) = (\lambda_1^2 - \lambda)(\lambda_2^2 - \lambda)(\lambda_3^2 - \lambda)$

$$\Rightarrow \quad I_C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \mathrm{tr}\, C$$
$$II_C = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2 = \frac{1}{2}\left[(\mathrm{tr}\, C)^2 - \mathrm{tr}\, C^2\right]$$
$$III_C = (\lambda_1\lambda_2\lambda_3)^2 = \det C$$

matrix invariants I_B, II_B, III_B: $\det(B - \lambda I) =: -\lambda^3 + I_B \lambda - II_B \lambda^2 + III_B$

$I_B = I_C$ since $\mathrm{tr}\, C = \mathrm{tr}\, B$; likewise $II_B = II_C$, $III_B = III_C$